Physical Applications:

Type 1: Mass & Density. (One dimensional Object)

First we consider a thin rod/wire with left end $x=a$ & right end $x=b$. As shown here →

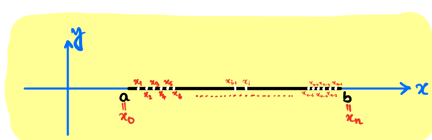
Consider the rod has constant density ρ.

Then the mass of the rod is, (density)·(length of the rod)

$$= \rho \cdot (b-a)$$

Now consider the situation, you have the same rod, but this time density of the rod is not constant.

Then along the rod, the density of the rod varies from point to point, and so we consider the density function ($\rho(x)$) that encodes the density of the rod at any point x.



We are partitioning the interval $[a,b]$ in $(n+1)$ subintervals & for each subinterval $[x_{i-1}, x_i]$ we choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$.

Then the mass m_i of the rod segment from x_{i-1} to x_i is approximated by $\quad m_i \approx \rho(x_i^*)(x_i - x_{i-1}) = \rho(x_i^*)\Delta x$

So the total mass of the rod can be approximated by

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} \rho(x_i^*)\Delta x$$

Riemann Sum

Then taking limit $n \to \infty$, we get

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)\Delta x = \int_a^b \rho(x)\,dx$$

Hence the mass-density formula of one-dimensional object is,

$$m = \int_a^b \rho(x)\,dx$$

$b \to$ right end of rod

$a \to$ left end of rod

, where $\rho(x)$ is the density function.

Q.1. Consider a thin rod oriented on the x-axis over the interval $[\pi/2, \pi]$. If the density of the rod is given by $\rho(x) = \sin x$, what is the mass of the rod?

Solⁿ:-

$$m = \int_a^b \rho(x)\,dx = \int_{\pi/2}^{\pi} \sin x\,dx = -\cos x \Big|_{\pi/2}^{\pi}$$

$$= (-\cos\pi) - (-\cos\pi/2)$$

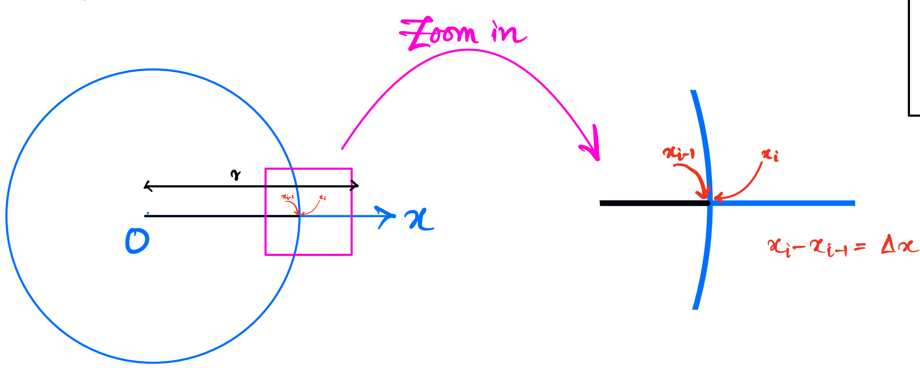
$= (-(-1)) - (-0)$

$= 1.$

Type 2: Mass & Density. (Two dimensional Object)

We now extend this concept to find the mass of a very thin disk of radius r, with density function $\rho(x)$. Orient the disk on the xy-plane & centre at the origin as shown below.

Note: This particular density function is called radial density function.



We partition the interval $[0, r]$ along the x-axis. Choose one partition $[x_{i-1}, x_i]$ & an arbitrary pt x_i^* in it.

Then we first consider the thin ring with inner radius x_{i-1} & outer radius x_i.

circular rod

So the area $A_i = \pi x_i^2 - \pi x_{i-1}^2$

$= \pi (x_i^2 - x_{i-1}^2)$

$$= \pi (x_i + x_{i-1})(x_i - x_{i-1})$$

$$= \pi (x_i + x_{i-1}) \Delta x$$

$$\approx \pi (x_i^* + x_i^*) \Delta x$$

$$= 2\pi x_i^* \Delta x$$

So, mass of the thin ring is $m_i = 2\pi x_i^* \rho(x_i^*) \Delta x$

Adding up all the mass of such rings we get,

$$m = \sum_{\substack{n \text{ partitions} \\ \text{of } [0,r]}} m_i = \underbrace{\sum_{\substack{n \text{ partitions} \\ \text{of } [0,r]}} 2\pi x_i^* \rho(x_i^*) \Delta x}_{\text{Riemann Sum}}$$

So, by taking $n \to \infty$ we get

$$m = \lim_{n \to \infty} \sum_{\substack{n \text{ partitions} \\ \text{of } [0,r]}} 2\pi x_i^* \rho(x_i^*) \Delta x$$

$$= \int_0^r 2\pi x \rho(x) \, dx$$

Q.2. Let $\rho(x) = 3x+2$ represent the radial density of a disk. Calculate the mass of a disc of radius 2.

Sol^n:- Here $r = 2$ & so,

$$m = \int_0^2 2\pi x\,(3x+2)\,dx = 2\pi \int_0^2 (3x^2+2x)\,dx$$

$$= 2\pi \left[3\cdot\frac{x^3}{3} + 2\cdot\frac{x^2}{2}\right]_0^2$$

$$= 2\pi \left[x^3 + x^2\right]_0^2$$

$$= 2\pi \left[(2^3+2^2) - (0^3+0^2)\right]$$

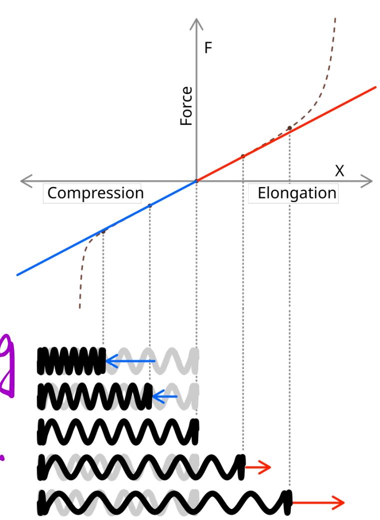
$$= 2\pi \left[(8+4) - 0\right]$$

$$= 24\pi.$$

Type 3: Work Done by a Force

Suppose a variable force $F(x)$ moves an object along the x-axis from position a to position b. Then the work done by the force $F(x)$ on the object is

$$W = \int_a^b F(x)\,dx$$



Hooke's Law for Springs: The force required to compress or elongate a spring from equilibrium position is $F(x) = kx$, $k>0$.

spring constant

Unit for Work	Length	Force	Work
Metric English	meters (m) feet (ft)	Newtons (N) Poundes (lb)	Joules (J) foot-pounds (ft-lb)

Q.3. A force of 30N is required to stretch a spring 5 cm from equilibrium. How much work is required to stretch the spring an additional 20 cm?

Sol^n:- Here, $F(x) = 30\,N$ ← metric unit system

$$x = 5\,cm = 0.05\,m$$

Then by Hooke's Law, $F(x) = kx$, $k \equiv$ spring constant

$$\Rightarrow 30 = k(0.05)$$

$$\Rightarrow k = \frac{30\,N}{(0.05)m} = 600\ N/m.$$

We want to find the Work done when the spring is stretched additional 20 cm.

So, $a = 5\,cm = 0.05\,m$

$$b = (5+20)\,cm = 25\,cm = 0.25\,m$$

Then, $W = \int_a^b F(x)\,dx = \int_{0.05}^{0.25} (600x)\,dx$

$$= 600 \int_{0.05}^{0.25} x \, dx$$

$$= 600 \left[\frac{x^2}{2}\right]_{0.05}^{0.25}$$

$$= 300 \left[(0.25)^2 - (0.05)^2\right]$$

$$= 300 \left[0.25 + 0.05\right]\left[0.25 - 0.05\right]$$

$$= 300 \, (0.30) \, (0.20)$$

$$= 18 \text{ Joules.}$$

Type 4 : Work Done in Pumping.

Working Rule:

1. Sketch & fix a reference frame.
2. Calculate the volume of a representative layer of liquid (specifically for water, weight density $= 62.4 \, lb/ft^3$)
3. Multiply volume with weight-density (weight/unit volume) of the liquid to get force.
4. Compute the distance of the representative layer.
5. Multiply force in 3 & distance in 4 to get the work need corresponding to the representative layer.

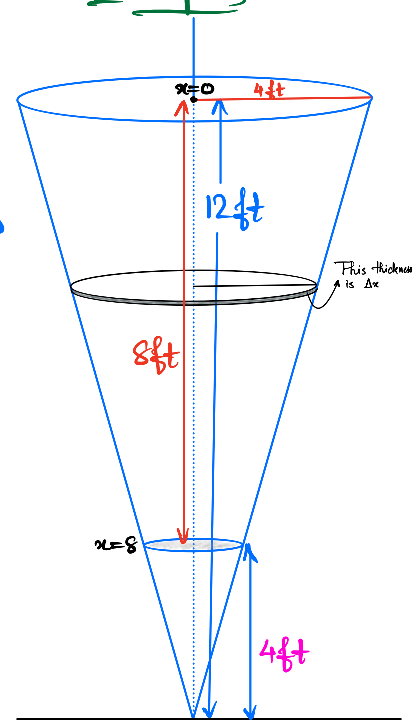
⑥ Sum the work for all layers.

⑦ Take $n \to \infty$ to get the resulting integral formula & then evaluate to get the exact work.

Q.4. A tank in the shape of an inverted cone, with height 12ft & base 4ft radius. The tank is full to start with & water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4ft. How much work is required to pump out that amount of water?

Sol^n: We need to partition the interval $[0, 8]$.

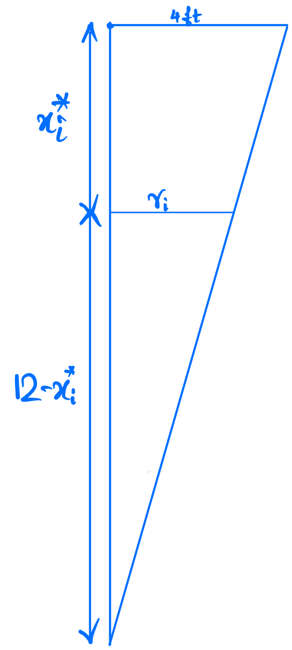
[Step 1]



We divide $[0, 8]$ into $(n+1)$ subintervals & choose an arbitrary point x_i^* in the subinterval $[x_{i-1}, x_i]$.

Now, corresponding to each such subinterval we can approximate the volume of a layer using a disc.

Now we just need to find the radius (r_i) of that representative layer of water.



From the properties of similar triangles we get,

$$\frac{r_i}{12-x_i^*} = \frac{4}{12} \Rightarrow \frac{r_i}{(12-x_i^*)} = \frac{1}{3}$$

$$\Rightarrow r_i = \frac{(12-x_i^*)}{3}$$

$$= 4 - \frac{x_i^*}{3}$$

Then the volume of the representative disc is,

$$V_i \approx \pi\left(4-\frac{x_i^*}{3}\right)^2 . \Delta x. \quad \text{[Step 2]}$$

Since the weight-density of water is 62.4 lb/ft^3.

So the force needed to lift each layer is approx.

$$F_i \approx 62.4\, V_i$$

$$= 62.4\pi\left(4-\frac{x_i^*}{3}\right)^2 \Delta x \quad \text{[Step 3]}$$

We need to lift each layer corresponding x_i^* ft.

[Step 4]

Therefore, the approximate work needed to lift each layer is $W_i = F_i \times x_i^*$

$$\approx 62.4\pi\, x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \qquad \text{[Step 5]}$$

Summing up the work for all such layers we get

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 62.4\pi\, x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \qquad \text{[Step 6]}$$

Now taking limit $n \to \infty$ we get

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 62.4\pi\, x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x$$

$$= \int_0^8 62.4\pi\, x \left(4 - \frac{x}{3}\right)^2 dx$$

$$= 62.4\pi \int_0^8 x \left(16 - \frac{8x}{3} + \frac{x^2}{9}\right) dx$$

$$= 62.4\pi \int_0^8 \left(16x - \frac{8x^2}{3} + \frac{x^3}{9}\right) dx$$

$$= 62.4\pi \left[16 \cdot \frac{x^2}{2} - \frac{8}{3} \cdot \frac{x^3}{3} + \frac{1}{9} \cdot \frac{x^4}{4}\right]_0^8$$

$$= 10{,}649.6\pi \approx 33{,}456.7$$

Hence it takes approximately 33,457 ft-lb of work to empty the tank to the desired level.